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Hodge Decomposition Along the Leaves of a Riemannian Foliation

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A Hodge decomposition theorem is proved for the leafwise Laplacian of a Riemannian foliation on a closed Riemannian manifold. © 1991 Academic Press, Inc.

1. Introduction and Results

Let $\mathcal{F}$ be a smooth foliation on a closed oriented manifold M. Let $T\mathcal{F}$ be its tangent bundle and $Q = TM/T\mathcal{F}$, the normal bundle. Let g_M be a Riemannian metric on M, which decomposes TM as an orthogonal direct sum $T\mathcal{F} \oplus T\mathcal{F}^\perp$, and canonically identifies $T\mathcal{F}^\perp$ with Q. This gives rise to the bigrading of the algebra of smooth differential forms $\Omega = \Omega_M$

$$\Omega^{u,v} = \Gamma(\Lambda^v T\mathcal{F}^* \otimes \Lambda^u Q^*) \cong \Gamma\Lambda^v T\mathcal{F}^* \otimes_{C^\infty(M)} \Gamma\Lambda^u Q^*. \tag{1.1}$$

The exterior derivative d decomposes as $d = d_{0,1} + d_{1,0} + d_{2,-1}$, where the double subscripts indicate the bidegrees of the bihomogeneous components. The spectral sequence associated to the filtration

$$F^r\Omega = \bigoplus_{u \geq r} \Omega^{u,\cdot} \tag{1.2}$$

satisfies then canonically $(E_0, d_0) \cong (\Omega, d_{0,1})$ (see, e.g., [1, 11]). A natural idea is to try to find a relation between the cohomology $E_1 = H(\Omega, d_{0,1})$ and the harmonic forms associated to the leafwise Laplacian Δ_0, the

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Laplacian canonically associated to $d_{0,1}$. If $\delta_{0,-1}$ denotes the formal adjoint of $d_{0,1}$, and $D_0 = d_{0,1} + \delta_{0,-1}$ the "Dirac operator along the leaves," then $\Delta_0 = D_0^2$. These operators are also closely related to the operators used in the study of the index problem in [4, 17], and in the study of $\mathscr{F}$-harmonic measures in [7]. In the spirit of Milgram and Rosenbloom [13], one has to study the heat flow associated to Δ_0, and its long-time behavior.

In [14] a similar study has been carried out for the basic Laplacian Δ_B acting on the subcomplex $\Omega_B \subset \Omega_M$ of basic forms of a Riemannian foliation, thereby reproving the Hodge decomposition theorem of [6, 12]. This succeeded because of formal ellipticity properties of Δ_B.

In contrast, the operator Δ_0 is defined on all forms but is only "elliptic along the leaves." A first step towards the desired decomposition theorem can be made as follows. First one constructs a chain of Hilbert space completions $\{H_{0,r}\}_{r \geqq 0}$ of Ω_M with $H_{0,r} \supset H_{0,r+1}$, and compatible extensions $\bar{d}_{0,1,r}$, $\bar{\delta}_{0,-1,r}$, $\bar{D}_{0,r} = \bar{d}_{0,1,r} + \bar{\delta}_{0,-1,r}$, which define self-adjoint extensions $\bar{\Delta}_{0,r} = \bar{D}_{0,r}^2 \colon \mathrm{Dom}(\bar{\Delta}_{0,r}) \subset H_{0,r} \to H_{0,r}$ of $\Delta_0 \colon \Omega_M \to \Omega_M$. These induce canonical continuous extensions $\bar{d}_{0,1,\infty}$, $\bar{\delta}_{0,-1,\infty}$, $\bar{D}_{0,\infty} = \bar{d}_{0,1,\infty} + \bar{\delta}_{0,-1,\infty}$ and finally $\bar{\Delta}_{0,\infty} = \bar{D}_{0,\infty}^2 \colon H_{0,\infty} \to H_{0,\infty}$ to the Fréchet space

$$H_{0,\infty} = \bigcap_{r \geqq 0} H_{0,r}. \tag{1.3}$$

The heat flows associated to the operators $\bar{\Delta}_{0,r}$ for $t \geqq 0$ yield semi-groups of bounded operators

$$H_{0,r} \xrightarrow{e^{-t\bar{\Delta}_{0,r}}} H_{0,r}. \tag{1.4}$$

For $t \to \infty$, these give rise to the following Hodge decomposition theorem.

THEOREM A. *Let $\mathscr{F}$ be a smooth foliation on a closed Riemannian manifold. Then the Laplacian Δ_0 along the leaves yields an orthogonal direct sum decomposition*

$$H_{0,\infty} \cong \ker \bar{\Delta}_{0,\infty} \oplus \overline{\mathrm{im}\, \bar{\Delta}_{0,\infty}} \cong \ker \bar{\Delta}_{0,\infty} \oplus \overline{\mathrm{im}\, \bar{d}_{0,1,\infty}} \oplus \overline{\mathrm{im}\, \bar{\delta}_{0,-1,\infty}}.$$

An abstract analogous result is proved in Section 2 in a formal functional analytic setting. In Section 3 this is applied to the situation at hand.

In the case of an ordinary closed Riemannian manifold (viewed as a one leaf foliation, with $T\mathscr{F} = TM$ and $Q = 0$), the Laplacian in question is the ordinary Laplacian $\Delta = \Delta_M$. Then the scalar Bochner–Weitzenböck formula (see, e.g., [9]), implies that the usual Sobolev norms are equivalent with the operator defined norms given by (2.1) below, so that $H_{0,\infty}$ is the usual infinite Sobolev space H_∞ of Ω_M. From the Sobolev Lemma it follows that $H_\infty \cong \Omega_M$, so that the decomposition of Theorem A implies the usual Hodge decomposition on forms. As noted by Roe, this construction and interpretation works equally well for the case of complete oriented Riemannian manifolds of bounded geometry, see [18].

A similar interpretation of $H_{0,\infty}$ can be given for Riemannian foliations. The corresponding result proved in Section 4 is as follows.

THEOREM B. *Let the situation be as in Theorem* A, *and assume $\mathscr{F}$ to be Riemannian and the metric bundle-like. Then the space $H_{0,\infty}$ is the space of elements in $L^2(\Omega_M)$ defined by L^2-Cauchy sequences $(\omega_m)_{m\in N}$ of smooth forms $\omega_m \in \Omega_M$, such that any sequence of smooth forms obtained from $(\omega_m)_{m\in N}$ by leafwise derivatives of any order of its local coefficient functions is L^2-Cauchy.*

Therefore, $H_{0,\infty}$ is formed by the elements in $L^2(\Omega_M)$ whose leafwise derivatives of any order exist and are also in $L^2(\Omega_M)$.

Some comments on leafwise cohomologies resulting from these considerations are finally made in Section 5.

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2. An Abstract Hodge Decomposition Theorem

We begin with an arbitrary complex Hilbert space H, and a symmetric (not necessarily bounded) operator S on H with dense domain $D \subset H$. Suppose that S verifies the hypothesis of Lemma 2.1 of Chernoff's paper [3], i.e., S maps D into itself, and there is a one-parameter group V_t of unitary operators on H such that $V_t D \subset D$, $V_t S = S V_t$ on D, and $(d/dt)\, V_t s = i S V_t s$ for $s \in D$. The definitions and results of this section are also valid for real Hilbert spaces, if the above conditions are satisfied by the corresponding complexifications.

Let $T = S^2$, which is a positive operator defined on D. For each integer $r \geqq 0$, let $H_r(T)$ denote the Hilbert space completion of D with respect to the scalar product

$$\langle s, s' \rangle_r = \sum_{i=0}^{r} \langle T^i s, s' \rangle \qquad \text{for} \quad s, s' \in D. \tag{2.1}$$

For the corresponding norms $\|\ \|_r$, we have

$$r \leqq r' \Rightarrow \|s\|_r \leqq \|s\|_{r'} \qquad \text{for all} \quad s \in D. \tag{2.2}$$

Thus we obtain the chain of continuous inclusions

$$H = H_0(T) \supset H_1(T) \supset H_2(T) \supset \cdots \supset H_\infty(T), \tag{2.3}$$

where

$$H_\infty(T) = \bigcap_{r \geqq 0} H_r(T) \tag{2.4}$$

is equipped with the obvious Fréchet topology.

Since the operator S is $\langle\ ,\ \rangle_r$-symmetric, by [3, Lemma 2.1] each power of S is essentially self-adjoint in $H_r(T)$. Let $\bar{T}_r$ be the closure of T in $H_r(T)$. By (2.2) we have

$$r \leqq r' \Rightarrow \mathrm{Dom}(\bar{T}_r) \supset \mathrm{Dom}(\bar{T}_{r'}), \tag{2.5}$$

and the diagram

$$\begin{array}{ccc} \mathrm{Dom}(\bar{T}_r) & \xrightarrow{\bar{T}_r} & H_r(T) \\ \cup & & \cup \\ \mathrm{Dom}(\bar{T}_{r'}) & \xrightarrow{\bar{T}_{r'}} & H_{r'}(T) \end{array} \tag{2.6}$$

is commutative.

From definition (2.1) it follows that

$$H_{r+2}(T) \subset \mathrm{Dom}(\bar{T}_r), \tag{2.7}$$

and the restriction

$$\bar{T}_r \colon H_{r+2}(T) \to H_r(T) \tag{2.8}$$

is a bounded operator. Moreover, the commutativity of (2.6) implies the commutativity of the diagram

$$\begin{array}{ccc} H_{r+2}(T) & \xrightarrow{\bar{T}_r} & H_r(T) \\ \cup & & \cup \\ H_{r+3}(T) & \xrightarrow{\bar{T}_{r+1}} & H_{r+1}(T). \end{array} \tag{2.9}$$

Hence the operators $\bar{T}_r$ define a continuous operator

$$\bar{T}_\infty \colon H_\infty(T) \to H_\infty(T). \tag{2.10}$$

By the spectral theorem and the positivity of $\bar{T}_r$, we have the semi-group of bounded operators

$$e^{-t\bar{T}_r} \colon H_r(T) \to H_r(T) \qquad \text{for} \quad t \geqq 0. \tag{2.11}$$

2.12. LEMMA. *The diagrams*

$$\begin{array}{ccc} H_r(T) & \xrightarrow{e^{-t\bar{T}_r}} & H_r(T) \\ \cup & & \cup \\ H_{r+1}(T) & \xrightarrow{e^{-t\bar{T}_{r+1}}} & H_{r+1}(T) \end{array}$$

are commutative.

Proof. Let $\mathscr{B}_r$ be the subspace of the space of analytic vectors s (see [20], for example), such that for all $t \geqq 0$ the series

$$\sum_{n=0}^{\infty} \frac{(-t\bar{T}_r)^n}{n!} s \tag{2.13}$$

is convergent in $H_r(T)$, and so its limit is $e^{-t\bar{T}_r}s$.

By (2.2) we have $\mathscr{B}_{r+1} \subset \mathscr{B}_r$. Then, by the commutativity of (2.6) and the density of $\mathscr{B}_{r+1}$ in $H_{r+1}(T)$, the result follows. ∎

Since $e^{-t\bar{T}_r}$ converges strongly (as $t \to \infty$) to the orthogonal projection P_0^r of $H_r(T)$ onto the kernel of $\bar{T}_r$ (by the spectral theorem), from (2.12) we obtain the commutativity of the diagram

$$\begin{array}{ccc} H_r(T) & \xrightarrow{P_0^r} & H_r(T) \\ \cup & & \cup \\ H_{r+1}(T) & \xrightarrow{P_0^{r+1}} & H_{r+1}(T). \end{array} \tag{2.14}$$

It follows that

$$\ker(\bar{T}_r) \supset \ker(\bar{T}_{r+1}). \tag{2.15}$$

From (2.12) we also have that the operators $e^{-t\bar{T}_r}$ define a continuous operator $e^{-t\bar{T}_\infty}: H_\infty(T) \to H_\infty(T)$.

For each $t > 0$

$$\psi_t(x) = \frac{e^{-tx} - 1}{x} \tag{2.16}$$

is a bounded continuous function on $[0, \infty)$, so we have the bounded operator $\psi_t(\bar{T}_r)$ on $H_r(T)$ defined by the spectral theorem. Using [5, p. 1199, Corollary 7], we obtain

$$\mathrm{Dom}(\bar{T}_r \psi_t(\bar{T}_r)) = H_r(\bar{T}). \tag{2.17}$$

It follows that for all $s \in H_r(T)$

$$s = e^{-t\bar{T}_r}s - \bar{T}_r \psi_t(\bar{T}_r)s. \tag{2.18}$$

These are the same arguments as in [10, Lemma A.4]. It is immediate that

$$\ker \bar{T}_r \perp \operatorname{im} \bar{T}_r \quad \text{in} \quad H_r(T). \tag{2.19}$$

Then (2.18) and (2.19) imply the following orthogonal direct sum decomposition

$$H_r(T) \cong \ker \bar{T}_r \oplus \overline{\operatorname{im} \bar{T}_r}. \tag{2.20}$$

From (2.5) and (2.6) we obtain further

$$\operatorname{im}(\bar{T}_r) \supset \operatorname{im}(\bar{T}_{r+1}). \tag{2.21}$$

Then, since $H_{r+1}(T)$ is $\| \ \|_r$-dense in $H_r(T)$, from (2.15), (2.20), and (2.21) we obtain that $\ker(\bar{T}_r)$ and $\overline{\operatorname{im}(\bar{T}_r)}$ are the $\| \ \|_r$-completions of $\ker(\bar{T}_{r+1})$ and $\operatorname{im}(\bar{T}_{r+1})$, respectively. But it is clear that on $\ker(\bar{T}_{r+1})$ we have $\| \ \|_{r+1} = \| \ \|_r$, and thus

$$\ker(\bar{T}_0) = \ker(\bar{T}_1) = \ker(\bar{T}_2) = \cdots = \ker(\bar{T}_\infty). \tag{2.22}$$

Therefore, since $\operatorname{im}(\bar{T}_\infty)$ is dense in each $\operatorname{im}(\bar{T}_r)$, the following fact results.

2.23. PROPOSITION. *Under the above conditions, we have the orthogonal direct sum decomposition*

$$H_\infty(T) \cong \ker(\bar{T}_\infty) \oplus \overline{\operatorname{im}(\bar{T}_\infty)}. \tag{2.24}$$

This result can be sharpened as follows.

2.25. PROPOSITION. *Under the same hypothesis, suppose that there is an integer* $r \geqq 0$ *such that* 0 *is not an accumulation point of* $\sigma(\bar{T}_r) - \{0\}$, *where* $\sigma(\bar{T}_r)$ *is the spectrum of* $\bar{T}_r$. *Then we have orthogonal direct sum decompositions*

$$H_{r'}(T) \cong \ker(\bar{T}_{r'}) \oplus \operatorname{im}(\bar{T}_{r'}) \qquad \textit{for all} \quad r' \geqq r, \tag{2.26}$$

and

$$H_\infty(T) \cong \ker(\bar{T}_\infty) \oplus \operatorname{im}(\bar{T}_\infty). \tag{2.27}$$

Proof. By (2.5) and the commutativity of (2.6) we have

$$r \leqq r' \Rightarrow \sigma(\bar{T}_r) \supset \sigma(\bar{T}_{r'}). \tag{2.28}$$

So, if the hypothesis is verified for some integer $r \geqq 0$, it is also verified for any integer $r' \geqq r$. Therefore, it is enough to prove (2.26) for $r' = r$.

Suppose that 0 is not an accumulation point of $\sigma(\bar{T}_r) - \{0\}$, and let $N = \inf(\sigma(\bar{T}_r) - \{0\}) > 0$. Then we have

$$\|\bar{T}_r s\|_r \geqq N \cdot \|s\|_r \qquad \text{for all} \quad s \in \ker(\bar{T}_r)^{\perp} \cap \mathrm{Dom}(\bar{T}_r). \tag{2.29}$$

Let $\bar{T}_r^{\perp}$ be the restriction of $\bar{T}_r$ to

$$\mathrm{Dom}(\bar{T}_r^{\perp}) = \ker(\bar{T}_r)^{\perp} \cap \mathrm{Dom}(\bar{T}_r). \tag{2.30}$$

$\bar{T}_r^{\perp}$ is a self-adjoint operator on $\ker(\bar{T}_r)^{\perp}$. Then by (2.29) and the spectral theorem, we have that $1/\bar{T}_r^{\perp}$ is a well-defined bounded operator on $\ker(\bar{T}_r)^{\perp}$.

Again using [5, p. 1199, Corollary 7] we obtain

$$\mathrm{Dom}(\bar{T}_r^{\perp}(1/\bar{T}_r^{\perp})) = \ker(\bar{T}_r)^{\perp}, \tag{2.31}$$

and

$$\bar{T}_r^{\perp} \circ (1/\bar{T}_r^{\perp}) = \mathrm{id}. \tag{2.32}$$

Therefore,

$$\ker(\bar{T}_r)^{\perp} = \mathrm{im}(\bar{T}_r^{\perp}) = \mathrm{im}(\bar{T}_r). \tag{2.33}$$

which implies (2.26).

To prove (2.27), it is enough to prove

$$\bigcap_{r' \geqq r} \mathrm{im}(\bar{T}_{r'}) = \mathrm{im}(\bar{T}_{\infty}). \tag{2.34}$$

By assuming this we obtain that $\mathrm{im}(\bar{T}_{\infty})$ is closed in $H_{\infty}(T)$.

To prove (2.34) we observe that clearly $\mathrm{im}(\bar{T}_{\infty})$ is contained in each $\mathrm{im}(\bar{T}_{r'})$. On the other hand, take $s \in H_{\infty}(T)$, so that for every $r' \geqq r$ there exists $s_{r'} \in \mathrm{Dom}(\bar{T}_{r'})$ such that $\bar{T}_{r'}(s_{r'}) = s$. Then we can suppose that $s_{r'} \in \ker(\bar{T}_{r'})^{\perp}$, obtaining that $s_{r'} = s_r$ for all $r' \geqq r$ and $s \in \mathrm{im}(\bar{T}_{\infty})$. ∎

Under the general hypothesis of this section we have the following lemmas.

2.35. Lemma. *For all integers $r \geqq 0$ we have*

$$\mathrm{Dom}(\bar{T}_r) \subset H_{r+1}(T). \tag{2.36}$$

Proof. Let s be in $\mathrm{Dom}(\bar{T}_r)$. Then there exists a sequence (s_n) in D

which is $\| \|_r$-convergent to s and (Ts_n) is $\| \|_r$-convergent in $H_r(T)$. For integers $n, m \geqq 0$ we have then

$$\begin{aligned}\|s_n - s_{n+m}\|_{r+1}^2 &= \sum_{i=0}^{r+1} \langle T^i(s_n - s_{n+m}), s_n - s_{n+m} \rangle \\ &= \|s_n - s_{n+m}\|_0^2 + \langle Ts_n - Ts_{n+m}, s_n - s_{n+m} \rangle_r \\ &\leqq \|s_n - s_{n+m}\|_r^2 + \|Ts_n - Ts_{n+m}\|_r \cdot \|s_n - s_{n+m}\|_r,\end{aligned}$$

and the last expression converges to zero as $n \to \infty$. Therefore, (s_n) is $\| \|_{r+1}$-Cauchy, which implies that $s \in H_{r+1}(T)$. ▮

2.37. Lemma. *For all $t \geqslant 0$ and all integers $r \geqq 0$ we have*

$$\operatorname{im}(e^{-t\bar{T}_r}) \subset H_\infty(T). \tag{2.38}$$

Proof. Consider the real functions $f(x) = x$ and $g(x) = e^{-tx}$. Then by [5, p. 1199, Corollary 7] we have

$$\operatorname{Dom}(f(\bar{T}_r)\, g(\bar{T}_r)) \subset \operatorname{Dom}((fg)(\bar{T}_r)) \cap \operatorname{Dom}(g(\bar{T}_r)) = H_r(T). \tag{2.39}$$

Hence, by (2.35) we obtain

$$\operatorname{im}(e^{-t\bar{T}_r}) \subset \operatorname{Dom}(\bar{T}_r) \subset H_{r+1}(T). \tag{2.40}$$

Therefore, by (2.12) we have

$$\operatorname{im}((e^{-t\bar{T}_r})^k) \subset H_{r+k}(T) \qquad \text{for all} \quad k > 0, \tag{2.41}$$

and since

$$e^{-t\bar{T}_r} = (e^{-(t/k)\bar{T}_r})^k, \tag{2.42}$$

the result follows. ▮

2.43. Corollary. *The operators $\bar{T}_r$ and $\bar{T}_\infty$ have the same eigenvalues, and all their eigenvectors are in $H_\infty(T)$.*

Proof. By the commutativity of (2.9) it is enough to prove that the eigenvectors of each operator $\bar{T}_r$ are in $H_\infty(T)$. But if s is an eigenvector of $\bar{T}_r$ and λ the corresponding eigenvalue, by the spectral theorem we have $e^{-\bar{T}_r}s = e^{-\lambda}s$, so $s \in H_\infty(T)$ by (2.37). ▮

2.44. Proposition. *Assuming the conditions of* (2.23), *and in addition there is some $r \geqq 0$ such that $H_r(T)$ is separable and the inclusion $H_{r+1}(T) \subsetneq H_r(T)$ is compact, then there exists a complete orthonormal* *system* (COS) $\{s_i\}$ *for* $H_r(T)$, *consisting of eigenvectors of* $\bar{T}_\infty$ *in* $H_\infty(T)$, *and if* $H_r(T)$ *is of infinite dimension, then the corresponding eigenvalues satisfy* $0 \leqq \lambda_1 \leqq \lambda_2 \leqq \cdots \uparrow \infty$ (*so they are of finite multiplicity*).

Proof. We can suppose that $H_r(T)$ is of infinite dimension. As in the classical situation (see, e.g., [8]), one inverts the Friedrichs extension $T^{\mathrm{F}}_{r,1}$ of the operator $\bar{T}_r + \mathrm{id}$ to obtain a bounded operator

$$G^{\mathrm{F}}_{r,1} = (T^{\mathrm{F}}_{r,1})^{-1}\colon H_r(T) \to D(\bar{T}_r) \cap H_{r+1}(T), \tag{2.45}$$

which composed with the compact inclusion into $H_r(T)$ yields a compact self-adjoint strictly positive operator on $H_r(T)$. Thus $G^{\mathrm{F}}_{r,1}$ has positive eigenvalues $\mu_1 \geqq \mu_2 \geqq \cdots \downarrow 0$ with corresponding eigenvectors $\{s_i\}$ constituting a COS for $H_r(T)$. Then it follows that $\bar{T}_r s_i = \lambda_i s_i$ with $\lambda_i = \mu_i^{-1} - 1$, satisfying $0 \leqq \lambda_1 \leqq \lambda_2 \leqq \cdots \uparrow \infty$. Moreover, $s_i \in H_\infty(T)$ by (2.43). ∎

Note that the hypothesis of (2.44) also implies the hypothesis of (2.25), so the orthogonal decompositions (2.26) and (2.27) also hold and $\bar{T}_r$ is a Fredholm operator (with index zero, since it is self-adjoint).

Proposition 2.44 is the abstract result generalizing the situation of the usual Laplacian associated to a closed Riemannian manifold, where the compactness of the Sobolev inclusions is given by Rellich's Lemma. It also has the following converse.

2.46. Proposition. *Under the hypothesis of* (2.23), *assume that* H *is separable and has a* COS$\{s_i\}$ *consisting of eigenvectors of* T, *such that the corresponding eigenvalues satisfy* $0 \leqq \lambda_1 \leqq \lambda_2 \leqq \cdots \uparrow \infty$. *Then each inclusion* $H_{r+1}(T) \subsetneq H_r(T)$ *is compact.*

Proof. From (2.1) we obtain for any $r \geqq 0$

$$\langle s, s' \rangle_r = \sum_{j=0}^{r} \langle \bar{T}^j_r s, s' \rangle \qquad \text{for} \quad s \in H_\infty(T) \text{ and } s' \in H_r(T). \tag{2.47}$$

Hence, by (2.43) each $H_r(T)$ has a COS $\{s_{r,i}\}$ given by

$$s_{r,i} = (\gamma_{r,i})^{-1/2}\, s_i, \qquad \text{where} \quad \gamma_{r,i} = \sum_{j=0}^{r} \lambda_i^j; \tag{2.48}$$

obtaining that

$$\langle s_{r,i}, s \rangle_r = \gamma_{r,i}^{1/2} \cdot \langle s_i, s \rangle \qquad \text{for all} \quad s \in H_r(T). \tag{2.49}$$

Let B be the unit ball in H_{r+1}. Given $\varepsilon > 0$ choose an integer $L > 0$ such that $L^{-1/2} < \varepsilon$. Since $0 \leqq \lambda_1 \leqq \lambda_2 \leqq \cdots \uparrow \infty$, there exists an integer $N > 0$

such that $\lambda_i \geqq L$ for all $i \geqq N$. Let Z be the space of elements $s \in H_{r+1}(T)$ so that $\langle s_i, s\rangle = 0$ for $i = 1, ..., N$. Then for all $s \in B \cap Z$, we have by (2.48) and (2.49)

$$\begin{aligned} 1 > \|s\|_{r+1}^2 &= \sum_{i=1}^{\infty} \gamma_{r+1,i} \cdot |\langle s_i, s\rangle|^2 \\ &= \sum_{i=1}^{\infty} (\gamma_{r,i} \cdot \lambda_i + 1) \cdot |\langle s_i, s\rangle|^2 \\ &\geqq L \cdot \|s\|_r^2, \end{aligned}$$

and thus $\|s\|_r < \varepsilon$. Now the argument follows as in the proof of Rellich's Lemma (see [19, Chapter III]). The unit ball of $H_{r+1}(T)/Z$ is precompact because Z is of finite codimension in $H_{r+1}(T)$, so it can be covered by finitely many balls of radius ε. Therefore, B can be covered by finitely many $\|\ \|_r$-balls of radius 2ε, and so B is precompact in $H_r(T)$. ∎

Let S_r be the closure of S in $H_r(T)$, which is self-adjoint. From the definition (2.1) it follows that

$$\operatorname{Dom}(\bar{S}_r) = H_{r+1}(T), \tag{2.50}$$

and

$$\begin{array}{ccc} H_{r+1}(T) & \xrightarrow{\bar{S}_r} & H_r(T) \\ \cup & & \cup \\ H_{r+2}(T) & \xrightarrow{\bar{S}_{r+1}} & H_{r+1}(T) \end{array} \tag{2.51}$$

is a commutative diagram of bounded operators. Thus the operators $\bar{S}_r$ define a continuous operator

$$\bar{S}_\infty : H_\infty(T) \to H_\infty(T). \tag{2.52}$$

It is easy to check that

$$\ker(\bar{S}_r) = \ker(\bar{T}_r) = \ker(\bar{T}_\infty) = \ker(\bar{S}_\infty), \tag{2.53}$$

and

$$\operatorname{im}(\bar{S}_r) \perp \ker(\bar{S}_r), \qquad \operatorname{im}(\bar{S}_\infty) \perp \ker(\bar{S}_\infty). \tag{2.54}$$

Therefore, from (2.20) and (2.23) we obtain the orthogonal decompositions

$$H_r(T) \cong \ker(\bar{S}_r) \oplus \overline{\operatorname{im}(\bar{S}_r)}, \tag{2.55}$$

and

$$H_\infty(T) \cong \ker(\bar{S}_\infty) \oplus \overline{\mathrm{im}(\bar{S}_\infty)}. \tag{2.56}$$

Propositions similar to (2.25), (2.44), and (2.46) hold replacing $\bar{T}_r$ and $\bar{T}_\infty$ by $\bar{S}_r$ and $\bar{S}_\infty$. But in this case there may be negative eigenvalues of $\bar{S}_\infty$, so the results corresponding to (2.44) and (2.46) have to be stated in the appropriate way.

3. Proof of Theorem A

We apply the results of the preceding section to a smooth foliation $\mathscr{F}$ on a closed Riemannian manifold (M, g_M). We have then the bigrading $\Omega^{u,v}$ of forms given by (1.2), and the decomposition $d = d_{0,1} + d_{1,0} + d_{2,-1}$. We can assume that M is oriented thus the formal adjoint δ of d can be given via the star operator $*$ on Ω (see, e.g., [15, Chap. IV]). It decomposes as $\delta = \delta_{0,-1} + \delta_{-1,0} + \delta_{-2,1}$ and for each (i, j) we have

$$\delta_{i,j} = (-1)^{n(r+1)+1} * d_{-i,-j} * \qquad \text{on} \quad \Omega^r,\ n = \dim M. \tag{3.1}$$

As explained in the Introduction, we then have the Dirac operator along the leaves

$$D_0 = d_{0,1} + \delta_{0,-1}, \tag{3.2}$$

and the Laplacian along the leaves

$$\Delta_0 = D_0^2 = d_{0,1}\,\delta_{0,-1} + \delta_{0,-1}\,d_{0,1}, \tag{3.3}$$

where the last equality follows from $d_{0,1}^2 = 0$, $\delta_{0,-1}^2 = 0$.

The complexification of the leafwise Dirac operator D_0 satisfies the hypothesis of Chernoff's Lemma 2.1 in [3]. This is obtained from [3, Corollary 1.4]. Then, with the notation of Section 2, we have the real Hilbert space $H_{0,r} = H_r(\Delta_0)$ and $H_{0,\infty} = H_\infty(\Delta_0)$. The operators D_0 and Δ_0 can be extended to continuous operators

$$\bar{D}_{0,\infty},\ \bar{\Delta}_{0,\infty} : H_{0,\infty} \to H_{0,\infty}, \tag{3.4}$$

yielding the orthogonal decompositions

$$H_{0,\infty} \cong \ker \Delta_{0,\infty} \oplus \overline{\mathrm{im}\ \bar{\Delta}_{0,\infty}} \cong \ker \bar{D}_{0,\infty} \oplus \overline{\mathrm{im}\ \bar{D}_{0,\infty}}. \tag{3.5}$$

Moreover, the spaces $\Omega^{u,v}$ are $\langle\ ,\ \rangle_{0,r}$-orthogonal to each other for each integer $r \geqq 0$, where $\langle\ ,\ \rangle_{0,r}$ denotes the scalar product of $H_{0,r}$. It follows that $\bar{D}_{0,\infty}$ can be decomposed as the sum of the continuous operators

$$\bar{d}_{0,1,\infty}, \bar{\delta}_{0,-1,\infty} : H_{0,\infty} \to H_{0,\infty}, \tag{3.6}$$

which are extensions of $d_{0,1}$ and $\delta_{0,-1}$, respectively. Since $\operatorname{im} d_{0,1}$, and $\operatorname{im} \delta_{0,-1}$ are $\langle\ ,\ \rangle_{0,r}$-orthogonal for each integer $r \geqq 0$, from (3.5) we obtain the orthogonal decomposition

$$H_{0,\infty} \cong \ker \Delta_{0,\infty} \oplus \overline{\operatorname{im} \bar{d}_{0,1,\infty}} \oplus \overline{\operatorname{im} \bar{\delta}_{0,-1,\infty}}. \tag{3.7}$$

This is the Hodge decomposition along the leaves of $\mathscr{F}$ stated in Theorem A.

4. Proof of Theorem B.

In the same context as in the preceding section, assume now moreover that the foliation $\mathscr{F}$ is Riemannian and the metric bundle-like. Let $p = \dim \mathscr{F}$, $q = \dim Q$, and $n = \dim M = p + q$.

We define a new connection on M by

$$\mathring{\nabla}_X Y = \pi_{\mathscr{F}} \nabla_X \pi_{\mathscr{F}} Y + \pi_Q \nabla_X \pi_Q Y \tag{4.1}$$

for vector fields X, Y, where $\nabla = \nabla^M$ is the Levi-Cività connection associated to g_M, and $\pi_{\mathscr{F}}$, π_Q the orthogonal projections of TM onto $T\mathscr{F}$, Q, respectively. The induced connection on ΛT^*M verifies

$$\mathring{\nabla}_X = (\nabla_X)_{0,0} \qquad \text{on} \quad \Omega_M, \text{ for all vector fields } X. \tag{4.2}$$

From Reinhart's characterization of bundle-like metrics in [16], it follows that $\nabla_Y Y = 0$ for any infinitesimal transformation Y of $\mathscr{F}$ of unit length, and orthogonal to $\mathscr{F}$. Thus $i_Y \nabla_Y \alpha = 0$ for all $\alpha \in \Omega^{0,\cdot}$. From this it is easy to obtain the following result using Koszul's formulas for d and δ (see, e.g., [15, Chap. IV]).

4.3. Proposition. *Let $E_1, \ldots, E_p$ be a (local) orthonormal frame of $T\mathscr{F}$ on a distinguished chart $U \subset M$, and $\alpha_1, \ldots, \alpha_p$ the dual co-frame of $T\mathscr{F}^* \mid U$. Then for any $\omega \in \Omega^{0,\cdot}(U)$*

$$d_{0,1}\omega = \sum_{i=1}^{p} \alpha_i \wedge \mathring{\nabla}_{E_i} \omega \tag{4.4}$$

$$\delta_{0,-1}\omega = -\sum_{i=1}^{p} i_{E_i} \mathring{\nabla}_{E_i} \omega. \tag{4.5}$$

For the following we can assume without loss of generality that M and $\mathscr{F}$ are oriented. Then the orientations of TM and $T\mathscr{F}$ induce an orientation of Q, yielding the star operators

$$*_{\mathscr{F}}: \Gamma\Lambda^v T^*\mathscr{F} \equiv \Omega^{0,v} \to \Omega^{0,p-v} \tag{4.6}$$

$$*_Q: \Gamma\Lambda^u Q^* \equiv \Omega^{u,0} \to \Omega^{q-u,0}, \tag{4.7}$$

with analogue properties to those of the usual star operator. A straightforward calculation gives the following lemma.

4.8. LEMMA. *We have according to the tensor expression* (1.1)

$$* = (-1)^{u(p-v)} *_{\mathscr{F}} \oplus *_Q : \Omega^{u,v} \to \Omega^{q-u,p-v}.$$

On a distinguished chart U we can find basic 1-forms $\beta_1, ..., \beta_q \in \Omega^{1,0}(U)$, i.e., satisfying $d_{0,1}\beta_j = 0$, such that for each $x \in U$ the $(\beta_j)_x$ form an oriented orthonormal basis of Q_x^*. This implies the decomposition

$$\Omega^{u,v}(U) = \Omega^{0,v}(U) \otimes \Lambda^u \left(\bigoplus_{j=1}^{q} \mathbb{R} \cdot \beta_j \right). \tag{4.9}$$

4.10. PROPOSITION. *With respect to the decomposition* (4.9), *we have* $D_0 = D_0 \otimes \mathrm{id}$ *on* $\Omega(U)$.

Proof. It is enough to prove $\delta_{0,-1} = \delta_{0,-1} \otimes \mathrm{id}$, which is reduced to a simple sign verification by using (3.1) and (4.8). ∎

With the same notation, each $\beta \in \Lambda(\bigoplus_{j=1}^q \mathbb{R} \cdot \beta_j)$ is supposed to be extended by zero to the whole manifold M. In this way, if $\beta \neq 0$ then it is not continuous on the boundary of U, but it defines an element in $L^2(\Omega_M)$.

4.11. LEMMA. *Let* $\alpha, \alpha' \in \Omega_c(U)$ *and* $\beta, \beta' \in \Lambda(\bigoplus_{j=1}^q \mathbb{R} \cdot \beta_j)$. *Then we have*

(i) $\beta \wedge *_Q \beta' = \pm \langle \beta, \beta' \rangle (1/\mathrm{Vol}(U)).\ \beta_1 \wedge \cdots \wedge \beta_q$

(ii) $\langle \alpha \wedge \beta, \alpha' \wedge \beta' \rangle = \pm \langle \alpha, \alpha' \rangle \langle \beta, \beta' \rangle (1/\mathrm{Vol}(U))$.

Proof. We can assume β and β' to be of the same degree. Then we have $\beta \wedge *_Q \beta' \in \mathbb{R} \cdot \beta_1 \wedge \cdots \wedge \beta_q$. Hence from (4.8) we obtain (i).
(ii) follows easily using (4.8) and (i). ∎

Now, for each integer $r \geq 0$ let $\mathring{\nabla}^r_{\mathscr{F}}: \Omega_M \to \Gamma(\otimes^r T\mathscr{F}^* \otimes \Lambda T^*M)$ be the operator defined by $\mathring{\nabla}$ in the usual way,

$$(\mathring{\nabla}^r_{\mathscr{F}} \omega)(X_1 \otimes \cdots \otimes X_r) = \mathring{\nabla}_{X_1} \cdots \mathring{\nabla}_{X_r} \omega \qquad \text{for} \quad X_i \in \Gamma T\mathscr{F}. \tag{4.12}$$

Also let further $\mathring{\nabla}^*_{\mathscr{F}}$ denote the formal adjoint of $\mathring{\nabla}_{\mathscr{F}} = \mathring{\nabla}^1_{\mathscr{F}}$.

Using arguments similar to those in [19, Chap. 2], (4.3) yields the "leafwise Bochner–Weitzenböck formula,"

$$\Delta_0 = \mathring{\nabla}_{\mathscr{F}}^* \mathring{\nabla}_{\mathscr{F}} + \mathring{R}_{\mathscr{F}} \qquad \text{on } \Omega_M^{0,\cdot}, \tag{4.13}$$

where $\mathring{R}_{\mathscr{F}}$ is an operator of order zero involving the curvature of $\mathring{\nabla}$.

It can be seen easily that in general (4.13) does not hold on the entire Ω_M. Nevertheless, we have the following result.

4.14. LEMMA. *For each integer $r \geqq 0$ there is a constant $c > 0$ such that if $\omega = \alpha \wedge \beta \in \Omega_c(U)$, with $\alpha \in \Omega_c^{0,\cdot}(U)$ and $\beta \in \Lambda(\bigoplus_{j=1}^q \mathbb{R} \cdot \beta_j)$, then we have*

(i) $\|\omega\|_{0,r} \leqq c \cdot \|\alpha\|_{0,r} \cdot \|\beta\|$,

(ii) $\|\alpha\|_{0,r} \cdot \|\beta\| \leqq c \cdot \|\omega\|_{0,r}$,

(iii) $\|\mathring{\nabla}_{\mathscr{F}}^r \omega\| \leqq c \cdot \sum_{i=0}^r \|\mathring{\nabla}_{\mathscr{F}}^i \alpha\| \cdot \|\beta\|$,

(iv) $\|\mathring{\nabla}_{\mathscr{F}}^r \alpha\| \cdot \|\beta\| \leqq c \cdot \sum_{i=0}^r \|\mathring{\nabla}_{\mathscr{F}}^i \omega\|$.

Proof. (i) and (ii) follow directly from (4.10) and (4.13). For any integer $m > 0$ and for all vector fields $X_1, ..., X_m$ we have

$$\mathring{\nabla}_{X_1} \cdots \mathring{\nabla}_{X_m} \omega = (\mathring{\nabla}_{X_1} \cdots \mathring{\nabla}_{X_m} \alpha) \wedge \beta + \sum_{k=1}^{m-1} \sum_{\sigma} (\mathring{\nabla}_{X_{\sigma_1}} \cdots \mathring{\nabla}_{X_{\sigma_k}} \alpha) \wedge L_\sigma \beta, \tag{4.15}$$

where $\sigma = (\sigma_1, ..., \sigma_k)$ ranges over the k-tuples of integers such that $1 \leqq \sigma_1 < \sigma_2 < \cdots < \sigma_k \leqq m-1$, and L_σ is a linear map of $\Lambda(\bigoplus_{j=1}^q \mathbb{R} \cdot \beta_j)$ to $\Omega(U)$, which is defined using the connection $\mathring{\nabla}$. By taking a smaller open subset of U if needed, we can consider the image of each L_σ as a subspace of $L^2(\Omega_M)$. Then, since $\Lambda(\bigoplus_{j=1}^q \mathbb{R} \cdot \beta_j)$ is of finite dimension, the operators L_σ are bounded. Therefore, (iii) follows directly from (4.15).

The proof of (iv) follows easily by induction using (4.11)(ii) and (4.15). ∎

4.16. PROPOSITION. *Each norm $\| \ \|_{0,r}$ is equivalent to the norm $\| \ \|'_{0,r}$ on Ω_M defined by*

$$\|\omega\|'_{0,r} = \left(\sum_{i=0}^r \|\mathring{\nabla}_{\mathscr{F}}^i \omega\|^2 \right)^{1/2}.$$

Proof. By (4.14) it is enough to prove that $\| \ \|_{0,r}$ is equivalent to $\| \ \|'_{0,r}$ on $\Omega^{0,\cdot}$, where it is a consequence of (4.13) by using a standard induction argument. ∎

Now Theorem B follows directly from Proposition 4.16.

5. Leafwise Cohomologies

Under the hypothesis of Theorem A, $(H_{0,\infty}, \bar{d}_{0,1,\infty})$ is a topological differential space, and its cohomology, $H(H_{0,\infty}, \bar{d}_{0,1,\infty})$, has the induced topology. We define the reduced cohomology of $(H_{0,\infty}, \bar{d}_{0,1,\infty})$, denoted $\mathscr{H}(H_{0,\infty}, \bar{d}_{0,1,\infty})$ (or simply $\mathscr{H}$), to be the quotient of $H(H_{0,\infty}, \bar{d}_{0,1,\infty})$ by the closure of the trivial subspace. Then from Theorem A we obtain a canonical isomorphism

$$\ker(\bar{\Delta}_{0,\infty}) \cong \mathscr{H}. \tag{5.1}$$

In the same way, the reduction $\mathscr{E}_1$ of E_1 is defined in [2] using the C^∞-topology of Ω. There is a canonical homomorphism of $\mathscr{E}_1$ to $\mathscr{H}$, and hence also to $K = \ker(\bar{\Delta}_{0,\infty})$.

The bigradation of Ω induces a bigradation on $H_{0,\infty}$, which defines a bigradation in both K and $\mathscr{H}$.

If M is oriented, by (3.1) the star operator $*$ induces isomorphisms

$$K^{u,v} \cong K^{q-u,p-v} \tag{5.2}$$

for all integers u and v. Thus, by (5.1) we also have duality isomorphisms

$$\mathscr{H}^{u,v} \cong \mathscr{H}^{q-u,p-v}. \tag{5.3}$$

If both M and $\mathscr{F}$ are oriented, then according to the tensor decomposition (1.1) we have the operators $*_{\mathscr{F}} \otimes \mathrm{id}$ and $\mathrm{id} \otimes *_Q$ on Ω. Then, assuming $\mathscr{F}$ to be Riemannian and the metric bundle-like, these operators also induce duality isomorphisms by (4.10)

$$K^{u,v} \cong K^{u,p-v}, \qquad K^{u,v} \cong K^{q-u,v}, \tag{5.4}$$

$$\mathscr{H}^{u,v} \cong \mathscr{H}^{u,p-v}, \qquad \mathscr{H}^{u,v} \cong \mathscr{H}^{q-u,v}. \tag{5.5}$$

Remark. In [17], J. Roe proves the following result for any smooth foliation $\mathscr{F}$ on a closed Riemannian manifold M. If $\alpha \in \Gamma \Lambda T^*\mathscr{F} \equiv \Omega_M^{0,\cdot}$, then $e^{-t\bar{\Delta}_{0,r}}\alpha \in \Omega_M^{0,\cdot}$ for any $t \geq 0$. When $\mathscr{F}$ is Riemannian and the metric is bundle-like, by (4.10) we also have $e^{-t\bar{\Delta}_{0,r}}(\Omega_M) \subset \Omega_M$. Considering the limit as $t \to \infty$, in general the orthogonal projection of $H_{0,\infty}$ on K does not preserve Ω_M. This can easily be seen for the Reeb foliation on S^3. Nevertheless we conjecture that it is true when $\mathscr{F}$ is Riemannian and the metric bundle-like, for in this case the distance between two leaves is locally constant, hence the limit as $t \to \infty$ of $e^{-t\bar{\Delta}_{0,r}}\alpha$ should not produce "transverse singularities" when α is smooth. If this conjecture were true, we would have the orthogonal decomposition

$$\Omega_M = \mathrm{Ker}(D_0) \oplus \overline{\mathrm{Im}(D_0)}, \tag{5.6}$$

and the canonical isomorphism

$$\mathscr{E}_1 \cong \mathrm{Ker}(D_0). \tag{5.7}$$

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